ON A QUESTION OF ERDŐS AND GRAHAM

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ABSTRACT. In this note, we sharpen work of Ulas to provide what is, in some sense, the minimal counterexample to a "conjecture" of Erdős and Graham about square values of products of disjoint blocks of consecutive integers.

1. INTRODUCTION

A remarkable paper of Erdős and Selfridge [2] completed a long-standing project of Erdős, showing that the product of consecutive integers cannot be a perfect power (and, in particular, a square). Referencing this problem, in Erdős and Graham [1] we find the following quote:

In the same spirit one could ask when the product of two or more disjoint blocks of consecutive integers can be a power. For example, if $A_1, \ldots, A_n$ are disjoint intervals each consisting of at least 4 integers then perhaps the product $\prod_{k=1} \prod_{a_k \in A_k} a_k$ is a nonzero square in only a finite number of cases.

In making such an assertion, one presumes that Erdős and Graham were guided by density arguments (which do indeed suggest finiteness for the integer points on the corresponding hypersurfaces). As noted in Ulas [5], however, these arguments can fail to hold in this situation; if the A_i's are taken to be blocks of precisely 4 integers, and if the number of such blocks is large enough, then the products take on square values infinitely often. In fact, Ulas suggests that there are likely infinitely many such blocks, in every case where the number of blocks is suitably large relative to the interval lengths. What seems to occur is that we have integral curves lying on our

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hypersurfaces; to predict or guarantee that such curves do in fact exist appears to be a hard problem.

In this short note, we will search for what might be termed minimal counterexamples to the proposal of Erdős and Graham (as one can observe from the quote above, it is probably unfair to characterize this as a conjecture). Along the way, we will sharpen and generalize the results of [5].

2. Some results

Let us, fixing a positive integer j and the j-tuple $(k_1, k_2, \ldots, k_j)$ of positive integers, consider the equation

$$\prod_{i=1}^{j} \prod_{l=0}^{k_i-1} (x_i + l) = y^2, \tag{2.1}$$

where the variables $x_1, x_2, \ldots, x_j$ are positive integers with the property that

$$x_s < x_t \text{ implies that } x_s + k_s \leq x_t. \tag{2.2}$$

We may clearly suppose, without loss of generality, that $j > 1$ (else we may appeal to the aforementioned theorem of Erdős-Selfridge) and that

$$2 \leq k_1 \leq k_2 \leq \cdots \leq k_j$$

(if $k_1 = 1$, then (2.1) has, trivially, infinitely many solutions).

Our two results are as follows. The first is a generalization of Theorem 1 of [5]. This result deals with situations omitted from consideration by Erdős and Graham; we include it for completeness.

Theorem 2.1. *If either $k_1 = 2$ or $(k_1, k_2) = (3, 3)$ then equation (2.1) has infinitely many solutions with (2.2).*

We also prove

Theorem 2.2. *If $j \geq 3$ and $k_i = 4$ for $1 \leq i \leq j$ then equation (2.1) has infinitely many solutions with (2.2).*

This latter result affirms a conjecture of Ulas (who deduced a like statement for $j = 4$ and $j \geq 6$). The families of examples we construct to show that (2.1) has infinitely many solutions with (2.2) in case $j = 3$ and $(k_1, k_2, k_3) = (4, 4, 4)$ are, we believe, minimal in j amongst counterexamples to the Erdős-Graham proposal.

3. Proof of Theorem 2.1

Let us begin by supposing that $k_1 = 2$. We choose the x_i's such that the product

$$\prod_{i=2}^{j} \prod_{l=0}^{k_i-1} (x_i + l)$$

is of the form $4m_1 m_2^2$, where $m_1 > 1$ is squarefree. In particular, if $j = 2$, $k_2 \in \{2, 3\}$, we take $x_2 = 3$, while, otherwise, we may choose the x_i $(i \geq 2)$ such that

$$\prod_{i=2}^{j} \prod_{l=0}^{k_i-1} (x_i + l) = m!$$

where $m = \sum_{i=2}^{j} k_i$ (note that $m!$ cannot be equal to a square, by Bertrand's Postulate). We thus find that (2.1) is satisfied precisely when there exists an integer y_1 for which

$$x_1(x_1 + 1) = m_1 y_1^2$$

or, equivalently,

$$(2x_1 + 1)^2 - m_1(2y_1)^2 = 1.$$

Since this equation has, for each squarefree m_1, infinitely many solutions in positive integers x_1 and y_1, we conclude that, if $j \geq 2$ and $k_1 = 2$, then (2.1) necessarily has infinitely many solutions.

Next, suppose that $(k_1, k_2) = (3, 3)$. If $j = 2$, as noted by K. R. S. Sastry, we may choose $x_1 = n$, $x_2 = 2n$, where n and m are positive integers satisfying

$$(n + 2)(2n + 1) = m^2$$

(see Guy [3]). As is well-known, there are infinitely many such solutions. We may therefore suppose that $j > 2$ and choose the x_i's such that the product

$$\prod_{i=3}^{j} \prod_{l=0}^{k_i-1} (x_i + l)$$

is of the form $m_1 m_2^2$, where $m_1 \neq 2$ is squarefree. To do this, we may take the x_i such that

$$\prod_{i=3}^{j} \prod_{l=0}^{k_i-1} (x_i + l) = m!$$

with $m = \sum_{i=3}^{j} k_i$,

$$x_2 = 2x_1 + 2 \quad \text{and} \quad x_1 = 3\,x_0.$$

Having a solution to equation (2.1) is thus equivalent to finding positive integers x_1 and y_1 satisfying

$$x_0(2x_0+1) = m!\,y_1^2$$

and so

$$(4x_0+1)^2 - 8\,m!\,y_1^2 = 1.$$

Since $m!/2$ is not a square (for $m > 2$), it follows that this equation has infinitely many solutions in integers x_0 and y_1. This completes our proof.

4. Proof of Theorem 2.2

As noted previously, Ulas derived Theorem 2.2 for $j = 4$ or $j \geq 6$. He observed a number of solutions in case $j = 3$ and strongly conjectured that there are infinitely many if $j = 5$. Since there exists a solution to equation (2.1) with

$$j = 2,\ k_1 = k_2 = 4,\ x_1 = 33,\ x_2 = 1680,$$

we may conclude as desired by showing that (2.1) has infinitely many solutions with (2.2), if $j = 3$. In fact, we will provide three infinite families of solutions in this case.

Let us begin by considering the Diophantine equation

$$u^2 - 3v^2 = -2. \tag{4.1}$$

The positive integral values of u that satisfy this equation are given by the recurrence

$$u_1 = 1,\ u_2 = 5,\ u_{n+1} = 4u_n - u_{n-1} \text{ for } n \geq 2.$$

There are thus infinitely many such solutions with $u \equiv 1 \mod 4$. If we additionally assume that $u \geq 265$ and set

$$(x_1, x_2, x_3) = \left(\frac{u-5}{4}, \frac{v-3}{2}, \frac{u+1}{2}\right),$$

then

$$x_1(x_3+2)x_2(x_2+3) = \frac{(u^2-25)(v^2-9)}{32} = \frac{(u^2-25)^2}{96}$$

and

$$(x_1+1)x_3(x_2+1)(x_2+2) = \frac{(u^2-1)(v^2-1)}{32} = \frac{(u^2-1)^2}{96},$$

whereby

$$\prod_{i=1}^{3}\prod_{l=0}^{3}(x_i+l)=\left(\frac{(u^2-1)(u+3)(u+7)(u^2-25)}{768}\right)^2.$$

The assumption that $u \geq 265$ guarantees that the x_i's satisfy (2.2).

Next, we note that (4.1) also has infinitely many solutions with $u \equiv -1 \mod 4$. For such a solution with $u \geq 3691$, we take

$$(x_1,x_2,x_3)=\left(\frac{u-7}{4},\frac{v-3}{2},\frac{u-7}{2}\right).$$

A little work shows that now

$$\prod_{i=1}^{3}\prod_{l=0}^{3}(x_i+l)=\left(\frac{(u^2-1)(u-3)(u-7)(u^2-25)}{768}\right)^2.$$

Our third family is also given by a recurrence. We now consider solutions to the equation

$$u^2-5v^2=4 \tag{4.2}$$

in odd integers u and v (so that $u \equiv 3 \mod 4$). We then take

$$(x_1,x_2,x_3)=\left(\frac{v-3}{2},\frac{u-7}{4},\frac{u+1}{2}\right)$$

and find that

$$\prod_{i=1}^{3}\prod_{l=0}^{3}(x_i+l)=\left(\frac{(u^2-9)(u+1)(u+5)(u^2-49)}{1280}\right)^2.$$

Notice that solutions to equation (4.2) satisfy $(u,v)=(L_{6n\pm 2},F_{6n\pm 2})$, where L_k and F_k denote the kth Lucas and Fibonacci numbers, respectively.

5. Concluding remarks

It is worth noting that the examples in [5] in case $j=4$ (and $k_i=4$ for $1 \leq i \leq 4$) grow polynomially (that is, the number of such examples with $\max\{x_i\}<X$ exceeds X^θ for some $\theta>0$), while those constructed here, for $j=3$ (and $k_i=4$) with $\max\{x_i\}<X$, are bounded in number by $c\log X$ for some constant c. It may be that this represents the true state of affairs for solutions to (2.1) in these instances, but it would appear to be most difficult to prove. More generally, it is possible that the behaviour of solutions to (2.1) is governed in some way by the size of $\sum_{i=1}^{j} 1/k_i$. There is no obvious heuristic that comes to mind to support this, however.

We suspect that the case $j = 3$, $k_1 = k_2 = k_3 = 4$ is, in some sense, minimal for (2.1) to have infinitely many solutions with (2.2). Indeed, we would guess that if $j = 2$ and $k_1 \geq 4$ then (2.1) has at most finitely many solutions with (2.2). The hypothesis that $k_1 \geq 4$ is certainly necessary here (even when we cannot apply Theorem 2.1) as it is easy to show that (2.1) has infinitely many solutions with $j = 2$ and $(k_1, k_2) = (3, 4)$ (as before, one can construct at least two families from recurrence sequences). An argument of P.G. Walsh (private communication) provides reasonable support (via the ABC conjecture) for the belief that the number of solutions to (2.1) with (2.2) if $j = 2, k_1 = k_2 = 4$ is finite.

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